LECTURE 5

$$C^m(\mathbb{R}^n)\big|_E$$

FOR E FINITE:

THE HEART OF THE MATTER

RECAP OF PREVIOUS TALKS:

GOOD THINGS FOLLOW

IF WE CAN PROVE

A MAIN THM

ON Γ's & σ's

IN THE SETTING OF

$C^{m,\omega}(\mathbb{R}^n)$.

In this lecture,
WE RECALL THE STATEMENT
OF THAT MAIN THM
& SKETCH ITS PROOF.

To avoid boring technicalities,
we restrict attention here
to $\omega(t) = \min(t, 1)$.

THE SETTING

Today we work in $C^m(\mathbb{R}^n)$.

$$J_x(F) = \left[\begin{array}{l} (m-1)^{\text{RST}} \text{ DEG. TAYLOR} \\ \text{POLY OF } F \text{ AT } x \end{array}\right]$$

$\mathcal{P}$ = VECTOR SPACE OF ALL $(m-1)^{\text{rst}}$ DEG. POLYS ON $\mathbb{R}^n$.

WE ARE GIVEN:

$\boxed{E \subset \mathbb{R}^n \text{ FINITE}}$

For each $x \in E$, $\ell \geq 0$, a symmetric convex set $\sigma_\ell(x) \subset \mathcal{P}$

For each $x \in E$, $\ell \geq 0$, $M > 0$, a (possibly empty) convex set $\Gamma_\ell(x, M) \subset \mathcal{P}$

The $T_\ell(x, M)$, $\sigma_\ell(x)$

are assumed to satisfy

the following :

$$\boxed{\Gamma_\ell(x, M) \subset \Gamma_\ell(x, M') \text{ if } M \leq M'.}$$

$$\boxed{|\partial^\alpha P(x)| \leq M \text{ (all } |\alpha| \leq m-1) \text{ for } P \in \Gamma_0(x, M)}$$

$$\boxed{|\partial^\alpha P(x)| \leq 1 \text{ (all } |\alpha| \leq m-1) \text{ for } P \in \sigma_0(x).}$$

$$\boxed{\begin{array}{l}\text{If } P \in \Gamma_\ell(x, M)\text{, then} \\ P + M\sigma_\ell(x) \subset \Gamma_\ell(x, C_1 M) \subset P + C_2 M \sigma_\ell(x).\end{array}}$$

WE SAY THAT

" $\Gamma_\ell(x, M)$ IS ESSENTIALLY A TRANSLATE OF $M\sigma_\ell(x)$."

NOTE THAT CONSTANTS C_1, C_2 ENTER THIS CONDITION.

Given $P \in \Gamma_\ell(x, M)$ $(\ell \geq 1)$

and

Given $y \in E$,

there exists $P' \in \Gamma_{\ell-1}(y, M)$

such that

$$|\partial^\alpha (P - P')(x)| \leq M|x-y|^{m-|\alpha|}$$

for $|\alpha| \leq m-1$.

WE SAY THAT "WE CAN MOVE TO NEARBY POINTS"

SIMILARLY,

$$\boxed{\begin{array}{l}\text{Given } P \in \sigma_\ell(x) \quad (\ell \geq 1) \\ \text{and} \\ \text{Given } y \in E \\ \text{there exists } P' \in \sigma_{\ell-1}(y) \\ \text{such that} \\ |\partial^\alpha (P - P')(x)| \leq |x-y|^{m-|\alpha|} \\ \qquad \text{for } |\alpha| \leq m-1.\end{array}}$$

WHITNEY ℓ-CONVEXITY

Given $P \in \sigma_\ell(x)$, $Q \in \mathcal{P}$, $0 < \delta \le 1$,

If P & Q satisfy

$$\left[\begin{array}{l} |\partial^\alpha P(x)| \le \delta^{m-|\alpha|} \\ \text{and} \\ |\partial^\alpha Q(x)| \le \delta^{-|\alpha|} \end{array}\right] \text{(all } |\alpha| \le m-1\text{)}$$

then

$$J_x(PQ) \in C_W \sigma_\ell(x)$$

NOTE THAT A CONSTANT C_W ENTERS.

UNDER THE ABOVE
ASSUMPTIONS,
WE WILL PROVE
THE FOLLOWING RESULT

MAIN THM:

Suppose $P_0 \in \Gamma_{\ell_*}(x_0, M_0)$

for a large enough ℓ_*

depending only on m & n.

Then there exists

$F \in C^m(\mathbb{R}^n)$

with the following properties.

$$\boxed{\|F\|_{C^m(\mathbb{R}^n)} \leq C_* M_0}$$

$$\boxed{\begin{array}{l} J_x(F) \in \Gamma_0(x, C_* M_0) \\ \text{for all } x \in E \end{array}}$$

$$\boxed{J_{x_0}(F) = P_0.}$$

Here, C_* depends only on $m, n,$ and on the constants C_1, C_2, C_w in our ASSUMPTIONS.

THAT'S OUR

MAIN THM

FROM LECTURE 4

(in the case $\omega(t) = \min(t, 1)$).

The rest of this lecture sketches its proof.

We keep the above SETTING & ASSUMPTIONS for the rest of the lecture.

CONVENTION:

CONSTANTS DENOTED

$c, C, C',$ ETC.

DEPEND ONLY ON THE CONSTANTS

C_1, C_2, C_W IN OUR ASSUMPTIONS,

and on m, n.

These symbols may denote different constants in different occurrences.

To prove our main thm,

WE CONSIDER

LOCAL VERSIONS

OF OUR INTERPOLATION

PROBLEM.

LOCAL INTERPOLATION PROBS.

Fix E, Γ's, σ's as above,

as well as $M_0 > 0$.

GIVEN

$\boxed{\begin{array}{l} \text{A cube } Q_0 \subset \mathbb{R}^n \\ \text{A point } x_0 \in E \cap 3Q_0 \\ \text{A poly } P_0 \in \mathcal{P}. \end{array}}$

The LOCAL INTERPOLATION PROB.

denoted $LIP(Q_0, x_0, P_0)$,

is to find an $F \in C^m(3Q_0)$

such that

$$\boxed{|\partial^\alpha F| \leq CM_0 \text{ on } 3Q \quad \text{for } |\alpha| = m}$$

$$\boxed{\begin{array}{l} J_x(F) \in \Gamma_0(x, CM_0) \\ \qquad \text{for all } x \in E \cap (1.01)Q \end{array}}$$

$$\boxed{J_{x_0}(F) = P_0.}$$

OUR MAIN THM

asserts that we can

solve $\mathrm{LIP}(Q^\circ, x_0, P_0)$,

for Q° = UNIT CUBE,

WHENEVER

$$P_0 \in \Gamma_{l_*}(x_0, M_0).$$

WE WILL PROVE THAT

ANY $LIP(Q^\circ, x^\circ, P^\circ)$

CAN BE SOLVED, PROVIDED

$P^\circ \in \Gamma_{\ell_*}(x^\circ, M_0)$.

WE WILL MEASURE

THE DIFFICULTY

OF A LOCAL INTERPOLATION PROB.

BY ATTACHING TO IT A

LABEL a.

Here, a LABEL $\mathcal{A}$ is any subset of

$$\mathcal{M} = \{\text{ALL MULTI-INDICES } \alpha = (\alpha_1, \dots, \alpha_n) \text{ OF ORDER } |\alpha| = \alpha_1 + \dots + \alpha_n \leq m-1\}$$

LABELS $\mathcal{a}$ COME WITH A (TOTAL) ORDER RELATION

$<$

[A VARIANT OF LEXICOGRAPHIC ORDER]

Roughly speaking, if $\mathcal{a} < \mathcal{B}$, then we expect that a typical LIP carrying the label $\mathcal{a}$ is easier than a typical LIP carrying the label $\mathcal{B}$.

If $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{B} < \mathcal{A}$.

Accordingly,

$\boxed{\text{The empty set } \emptyset \text{ labels the hardest interpolation problems}}$

and

$\boxed{\text{The set } \mathcal{M} = \{\alpha : |\alpha| \le m-1\} \text{ labels the easiest ones.}}$

A given interpolation problem

may carry

MORE THAN ONE LABEL.

WE FIRST EXPLAIN

$$\boxed{\text{HOW TO ATTACH A LABEL TO } LIP(Q_0, x_0, P_0),}$$

then discuss

$$\boxed{\text{How to use LABELS TO REDUCE HARD } LIP\text{'s TO EASIER ONES, thus proving our MAIN THM.}}$$

How to Attach Labels to Local Interpolation Probs.

Definitions & Simplest Properties

TO ASSIGN A LABEL

TO $LIP(Q_0, x_0, P_0)$,

WE EXAMINE THE

SIZE and SHAPE OF THE

$\mathcal{J}_\ell(x_0)$.

WE USE THE FOLLOWING

DEFINITION.

Suppose we are GIVEN

$\sigma \subset \mathcal{P}$ a symmetric convex set

$\mathcal{A} \subset \mathcal{M}$ a label

$x \in \mathbb{R}^n$ a point

$\delta, C_B > 0$, real numbers.

An $(\mathcal{A}, \delta, C_B)$ - BASIS FOR σ AT x

is a family of polys $P_\alpha \in \mathcal{P}$ indexed by $\alpha \in \mathcal{A}$, such that

$$\boxed{\begin{array}{l} \bullet\ \partial^\beta P_\alpha(x) = \delta_{\beta\alpha} \quad (\beta, \alpha \in \mathcal{A}) \\ \qquad \text{KRONECKER DELTA} \\ \bullet\ |\partial^\beta P_\alpha(x)| \leq C_B\, \delta^{|\alpha|-|\beta|} \quad (\alpha \in \mathcal{A}, \beta \in \mathcal{M}) \\ \bullet\ \delta^{m-|\alpha|} P_\alpha \in C_B\, \sigma \quad (\alpha \in \mathcal{A}) \end{array}}$$

RULE FOR ATTACHING A LABEL $\mathcal{A}$ TO $LIP(Q_0, x_0, P_0)$.

WE CAREFULLY PICK AN INTEGER CONST. $\ell(\mathcal{A})$ (DETERMINED BY $\mathcal{A}$, m, n) AND REAL CONSTS. $C_S(\mathcal{A})$, $C_B(\mathcal{A})$ (DETERMINED BY $\mathcal{A}$, m, n, C_1, C_2, C_W).

Let δ_{Q_0} = SIDELENGTH OF Q_0.

Then

$$\boxed{\text{WE ATTACH THE LABEL } \mathcal{A} \text{ TO } LIP(Q_0, x_0, P_0)}$$

$$\Longleftrightarrow$$

$$\boxed{\sigma_{\ell(\mathcal{A})}(x_0) \text{ has an } (\mathcal{A}, C_S(\mathcal{A})\delta_{Q_0}, C_B(\mathcal{A})) \text{ basis at } x_0.}$$

The label a tells us how much "ROOM TO MANEUVER" we have inside $\Gamma_{\ell}(x_0, M_0)$.

MORE PRECISELY, WE HAVE THE FOLLOWING RESULT.

LEMMA: Suppose $LIP(Q_0, x_0, P_0)$ carries a label $\mathcal{A}$, with $P_0 \in \Gamma_{\ell(\mathcal{A})}(x_0, C'M_0)$.

Let $\xi_\alpha \in \mathbb{R}$ $(\alpha \in \mathcal{A})$ satisfy

$$|\xi_\alpha| \leq C'' M_0 \delta_{Q_0}^{m-|\alpha|} \text{ for } \alpha \in \mathcal{A}.$$

Then there exists

$$\boxed{\begin{array}{l} \tilde{P} \in \Gamma_{\ell(\mathcal{A})}(x_0, C''' M_0) \text{ such that} \\ \partial^\alpha \tilde{P}(x_0) = \partial^\alpha P_0(x_0) + \xi_\alpha \text{ for } \alpha \in \mathcal{A}. \end{array}}$$

The LEMMA follows trivially from the definitions, together with the fact that

"$\Gamma_\ell(x, M)$ IS ESSENTIALLY A TRANSLATE OF $M\sigma_\ell(x)$".

SO LABELS TELL US THE DIRECTIONS IN WHICH $\Gamma_\ell(x_0, CM_0)$ IS

LONG.

RECALL: Those directions play a crucial rôle in interpolation problems.

REMARKS ON LABELS

- An $(\mathcal{A}, \delta, C_B)$-basis is a family of polys $(P_\alpha)_{\alpha \in \mathcal{A}}$ indexed by $\mathcal{A}$, such that ...

If $\mathcal{A} = \phi$, then no polys are required.

Therefore,

$$\boxed{\text{EVERY } (LIP(Q_\circ, x_\circ, P_\circ)) \text{ CARRIES THE LABEL } \phi}$$

- Suppose $\mathcal{A}' \subset \mathcal{A}$,

and suppose $(P_\alpha)_{\alpha \in \mathcal{A}}$

is an $(\mathcal{A}, \delta, C_B)$-basis for σ.

Then (trivially), $(P_\alpha)_{\alpha \in \mathcal{A}'}$

is an $(\mathcal{A}', \delta, C_B)$-basis for σ.

Thanks to this trivial observation

(and a couple of others),

we learn the following:

Every LIP(...) that

carries the label a

also carries the label a'

for every $a' \subset a$.

The RULE FOR ATTACHING A LABEL a TO $LIP(Q_0, x_0, P_0)$ MENTIONS Q_0 AND x_0, BUT NOT P_0.

So the assignment of a label a to $LIP(Q_0, x_0, P_0)$ IS INDEPENDENT of P_0.

(That will be important later.)

LET'S RECALL WHERE WE STAND

- WE WANT TO PROVE OUR MAIN THM
- WE'VE DEFINED $LIP(Q_0, x_0, P_0)$
- WE'VE SEEN THAT OUR MAIN THM asserts that $LIP(Q_0, x_0, P_0)$ can always be solved, provided $P_0 \in \Gamma_{\ell_*}(x_0, M_0)$.

- WE MEASURE THE DIFFICULTY OF $LIP(Q_0, x_0, P_0)$ BY ATTACHING LABELS α.

- WE'VE SEEN THE DEFINITION OF A LABEL, AND THE RULE THAT TELLS US WHEN WE MAY ATTACH A GIVEN LABEL α TO A GIVEN $LIP(Q_0, x_0, P_0)$.

- OUR PLAN IS TO USE LABELS TO REDUCE HARD LIP(...) TO EASIER ONES.

- LABELS COME WITH A (TOTAL) ORDER RELATION $<$.

ROUGHLY SPEAKING, A TYPICAL LIP(...) THAT CARRIES LABEL $\mathcal{A}$ IS EASIER THAN A TYPICAL LIP(...) THAT CARRIES LABEL $\mathcal{B}$, IF $\mathcal{A} < \mathcal{B}$.

- If $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B} < \mathcal{A}$.

So $\mathcal{M}$ is MINIMAL

and $\emptyset$ is MAXIMAL

under $<$.

ENOUGH REVIEW!

LET'S USE LABELS TO PROVE THE MAIN THM.

By INDUCTION ON THE LABEL a (with respect to $<$) we will prove the following.

MAIN LEMMA FOR $\mathcal{a}$:

Let $LIP(Q_0, x_0, P_0)$ be a Local Interpolation Prob. that carries the label $\mathcal{a}$.

Suppose $P_0 \in \Gamma_{\ell(\mathcal{a})}(x_0, M_0)$.

Then $LIP(Q_0, x_0, P_0)$ has a solution.

RECALL, A SOLUTION OF

$LIP(Q_0, x_0, P_0)$

IS A FUNCTION

$F \in C^m(3Q_0)$

such that

$$\boxed{\begin{array}{l} |\partial^\alpha F| \leq CM_0 \text{ on } 3Q_0 \text{ for } |\alpha| = m \\ J_x(F) \in \Gamma_0(x, CM) \text{ for } x \in E \cap (1.01)Q_0 \\ J_{x_0}(F) = P_0. \end{array}}$$

BECAUSE **EVERY** $LIP(\ldots)$

CARRIES THE LABEL ϕ,

THE MAIN LEMMA for ϕ

TELLS US THAT ANY

$LIP(Q_0, x_0, P_0)$ WITH

$P_0 \in \Gamma_{\ell(\phi)}(x_0, M_0)$ has a

solution.

That's precisely what we had to prove in order to establish our

MAIN THM

(WITH $\ell_* := \ell(\phi)$).

$$\boxed{\text{SO EVERYTHING COMES DOWN TO PROVING THE MAIN LEMMA FOR } a \text{ (any } a\text{).}}$$

Proof of the Main Lemma for a

by

Induction on a

(with Respect to $<$)

THE BASE CASE $\mathcal{A} = \mathcal{M}$

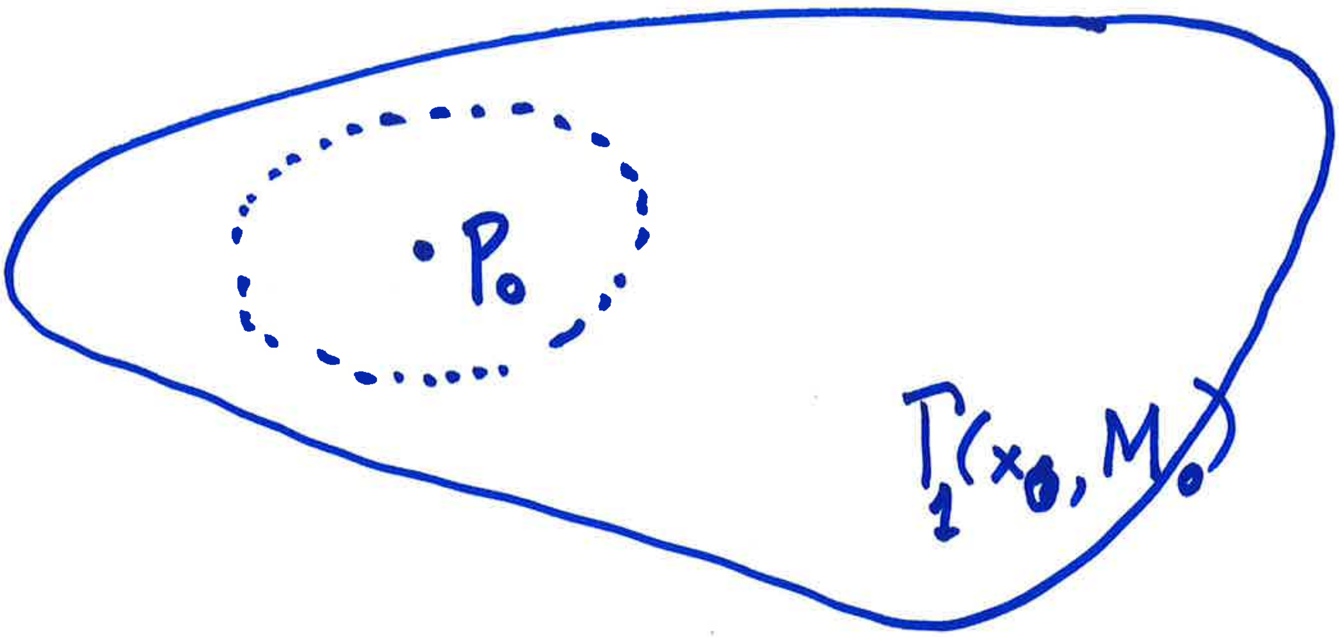
WE PROVE THE MAIN LEMMA for $\mathcal{M}$.

Suppose $LIP(Q_0, x_0, P_0)$ carries the label $\mathcal{M}$,

and suppose $P_0 \in \Gamma_1(x_0, M_0)$.

Because $LIP(\dots)$ carries the label $\mathcal{M}$, there is "room to maneuver" in <u>all</u> directions within $\Gamma_1(x_0, CM_0)$.

Therefore, $\Gamma_1(x_0, M_0)$ Contains an ellipsoid about P_0, as in the PICTURE



RECALL THAT GIVEN ANY

$P \in \Gamma_1(x_0, M_0)$

THERE EXISTS $\tilde{P} \in \Gamma_0(y, M_0)$

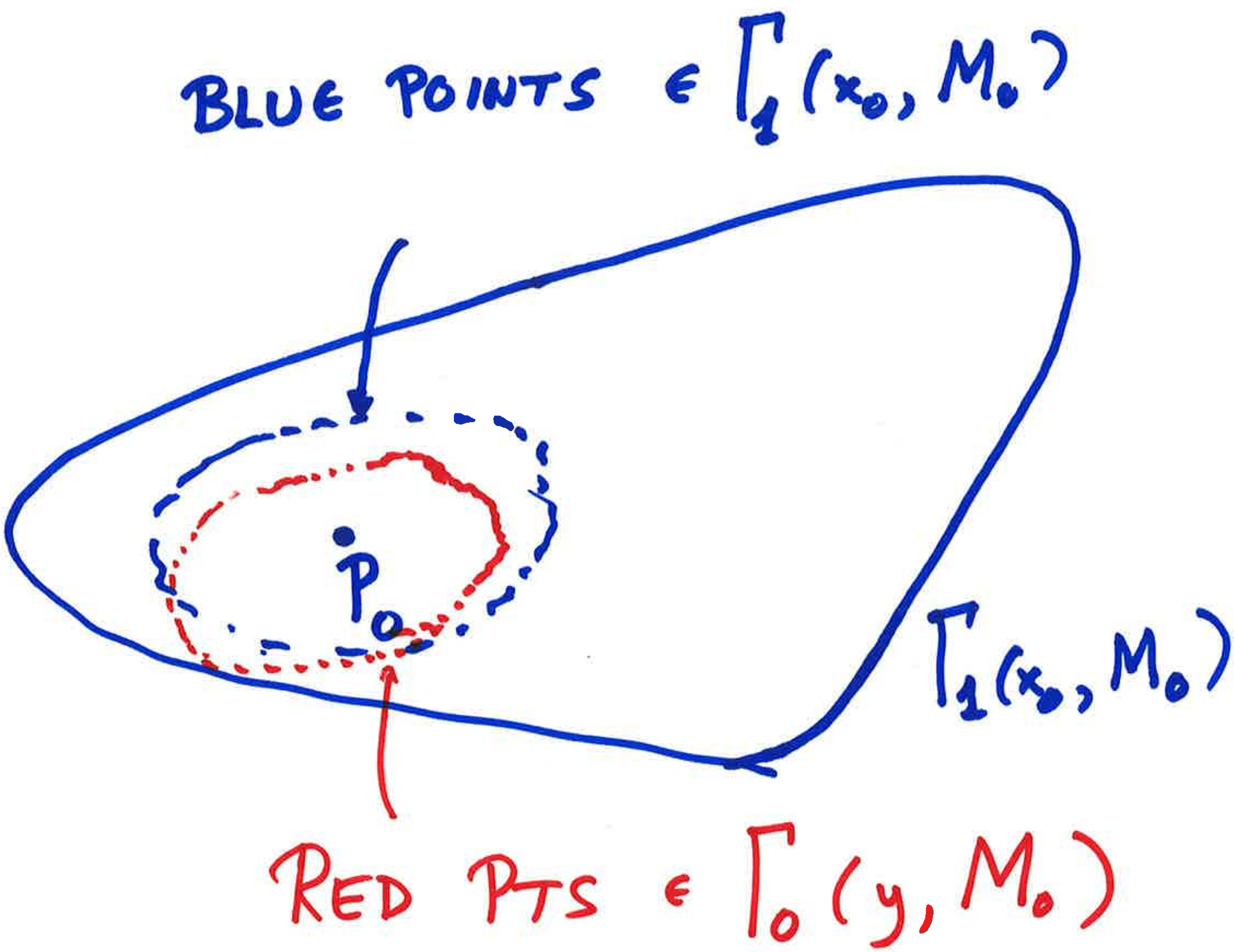
S.T.

$$|\partial^\alpha(\tilde{P} - P)(x_0)| \leq M|x_0 - y|^{m-|\alpha|}$$

(all α, any $y \in E$).

WE APPLY THIS REMARK TO ALL THE P IN THE DOTTED ELLIPSOID IN THE PICTURE.

(BLUE)



($y \in E$ NEAR x_0)

[NEAR EVERY BLUE DOT
LIES A RED DOT]

BECAUSE $\Gamma_0(y, M_0)$ IS CONVEX,

IT FOLLOWS THAT

$$\boxed{P_0 \in \Gamma_0(y, M_0)}$$

for all $y \in E$ "close" to x_0.

In particular, this holds for all $y \in E \cap (1.01) Q_0$.

Therefore, to solve

$LIP(Q_0, x_0, P_0)$,

we may simply take

$$F \equiv P_0.$$

Let's check that it works.

$$\boxed{|\partial^\alpha F| \leq C M_0 \text{ on } 3Q_0 \text{ for } |\alpha| = m}$$

In fact, for $|\alpha| = m$, we have $\partial^\alpha F \equiv 0$ because F is an $(m-1)^{\text{rst}}$ degree poly.

Next, note that

$J_x(F) = J_x(P_0) = P_0$ for all x,

hence

$$\boxed{J_{x_0}(F) = P_0}$$

and

$$\boxed{J_x(F) \in \Gamma_0(x_0, M_0) \text{ for all } x \in E \cap (1.01)Q_0}$$

as we just saw.

So, as promised,

in the

BASE CASE $a = m$

we can solve $LIP(Q_0, x_0, P_0)$

simply by taking $F = P_0$.

$$\boxed{\text{Thus, the MAIN LEMMA FOR } a \text{ holds for } a = m.}$$

PROOF OF THE MAIN LEMMA:

INDUCTION STEP

Fix a label a.

$\boxed{\text{ASSUME THAT THE MAIN LEMMA FOR } \hat{a} \text{ HOLDS WHENEVER } \hat{a} < a.}$

$\boxed{\text{UNDER THIS ASSUMPTION, WE PROVE THE MAIN LEMMA FOR } a.}$

THAT WILL:

$\boxed{\text{COMPLETE THE INDUCTION ON } a}$

$\boxed{\text{PROVE THE MAIN LEMMA FOR ALL } a}$

$\boxed{\text{PROVE TODAY'S MAIN THM}}$

$\boxed{\text{PROVE ALL THE THMS FROM TALK 3}}$

There are

2 CASES

for the proof of the

INDUCTION STEP:

Either

a IS <u>MONOTONIC</u>

The IMPORTANT CASE

or it <u>ISN'T</u>

The TRIVIAL CASE

RECALL, $\mathcal{A}$ IS A SET OF MULTI-INDICES OF ORDER $\leq m-1$.

$\mathcal{A}$ IS CALLED "MONOTONIC" if the following holds:

$$\boxed{\begin{array}{l}\text{Let } \alpha \in \mathcal{A},\ \beta \in \mathcal{M}. \\ \text{If } \alpha+\beta \in \mathcal{M}, \text{ then } \alpha+\beta \in \mathcal{A}.\end{array}}$$

WHY THE NON-MONOTONIC CASE IS ESSENTIALLY TRIVIAL

LEMMA:

Let $\mathcal{a}$ be a label,

and let α, β be multi-indices.

Suppose $\alpha \in \mathcal{a}$

and $\alpha + \beta \in \mathcal{M} \setminus \mathcal{a}$.

Then any

LIP(...) that carries

the label $\mathcal{a}$

also carries the label

$\hat{\mathcal{a}} = \mathcal{a} \cup \{\alpha + \beta\}$.

$$\boxed{\text{NOTE: } \hat{\mathcal{a}} < \mathcal{a}}$$

The LEMMA immediately implies our INDUCTION STEP in the NON-MONOTONIC CASE.

Indeed, by INDUCTION HYP., the MAIN LEMMA holds for $\hat{a}$.

That is, any $LIP(Q_0, x_0, P_0)$ that carries the label $\hat{a}$ and satisfies $P_0 \in \Gamma_{\ell(\hat{a})}(x_0, M_0)$

has a solution.

Applying the LEMMA,

we see that any $LIP(Q_0, x_0, P_0)$

that carries the label a

and satisfies

$$P_0 \in \Gamma_{\ell(\hat{a})}(x_0, M_0)$$

has a solution.

We pick the integer constants

$\ell(a)$ so that

$$\ell(\hat{a}) \leq \ell(a)$$

for $\hat{a} < a$,

and therefore

$$\Gamma_{\ell(a)}(x_0, M_0) \subset \Gamma_{\ell(\hat{a})}(x_0, M_0).$$

Consequently,

any $LIP(Q_0, x_0, P_0)$

that carries the label a

and satisfies $P_0 \in \Gamma_{\ell(a)}(x_0, M_0)$

has a solution.

That's the MAIN LEMMA for a.

So the INDUCTION STEP

for non-MONOTONIC $\mathcal{A}$

is reduced to the above LEMMA.

We'll omit the proof of the Lemma, & just remark that it uses the Whitney t-CONVEXITY of the sets $\sigma_\ell(x_0)$.

We are done with the

NON-MONOTONIC CASE.

The Induction Step in the Monotonic Case

- Fix $a \notin \mathcal{M}$ MONOTONIC
- Assume MAIN LEMMA for $\hat{a}$ (all $\hat{a} < a$)
- Fix $LIP_0 = LIP(Q_0, x_0, P_0)$.
- Assume LIP_0 CARRIES LABEL a but NO LABEL $\hat{a} < a$.
- Assume $P_0 \in \mathcal{T}_{\ell(a)}(x_0, M_0)$.

$\boxed{\text{UNDER THE ABOVE ASSUMPTIONS, WE WILL SOLVE } LIP_0.}$

That's the induction step in the monotoniz case.

THE PLAN

WE WILL SOLVE

$$LIP_0 = LIP(Q_0, x_0, P_0)$$

AS FOLLOWS.

1 WE CUT UP $3Q_0$ INTO SUBCUBES Q_ν $(\nu = 1, \cdots, \nu_{MAX})$ BY A CALDERÓN-ZYGMUND DECOMPOSITION

(2) $\boxed{\begin{array}{l}\text{For each } Q_\nu \text{, we} \\ \text{find a nearby point } x_\nu \in E \\ \text{and a poly } P_\nu \in \Gamma_{\ell(Q)-1}(x_\nu, M_0)\end{array}}$

Our Calderón-Zygmund STOPPING RULE guarantees that such an x_ν exists.

We can find such a P_ν close to P_0 (in $LIP(Q_0, x_0, P_0)$) by our BASIC ASSUMPTIONS on Γ's.

3) $\boxed{\text{For each } \nu, \text{ we consider the Local Interpolation Prob. } LIP(Q_\nu, x_\nu, P_\nu).}$

Our C-Z rule guarantees that the above problem satisfies the hypotheses of the MAIN LEMMA for $\hat{a}_\nu$ for some $\hat{a}_\nu < a$

(except in trivial cases, e.g. $E \cap 3Q_\nu = \emptyset$)

The INDUCTION HYP.

(MAIN LEMMA for any $\hat{a} < a$)

4) tells us that

$LIP(Q_\nu, x_\nu, P_\nu)$

has a solution $F_\nu \in C^m(3Q_\nu)$,

except in TRIVIAL CASES

E.G. $E \cap 3Q_\nu = \emptyset$.

IN THE TRIVIAL CASES,
WE EASILY CONSTRUCT
5) A SOLUTION $F_\nu \in C^m(3Q_\nu)$
OF $LIP(Q_\nu, x_\nu, P_\nu)$.

WE NOW HAVE A SOLUTION
OF
$LIP(Q_\nu, x_\nu, P_\nu)$
for ALL ν.

WE INTRODUCE A

WHITNEY PARTITION

OF UNITY

(6) $$1 = \sum_\nu \theta_\nu \quad \text{on } 3Q_0,$$

where

$$\operatorname{supp} \theta_\nu \subset (1.01) Q_\nu$$

and

$$|\partial^\alpha \theta_\nu| \leq C \delta_{Q_\nu}^{-|\alpha|} \quad (|\alpha| \leq m).$$

(δ_{Q_ν} = SIDELENGTH of Q_ν)

FINALLY, USING THE WHITNEY PARTITION OF UNITY, WE

7) PATCH TOGETHER THE LOCAL SOLUTIONS F_ν INTO (WHAT WE HOPE IS) A GLOBAL SOLUTION

$$F \equiv \sum_\nu \theta_\nu F_\nu \text{ on } 3Q_0 .$$

That's the PLAN.

LET'S EXECUTE the Plan.

HERE GOES ...

STEP 1:

WE MAKE A CALDERÓN-ZYGMUND DECOMPOSITION OF $3Q_0$ by SUCCESSIVELY "BISECTING" UNTIL WE ARE HAPPY.

The Calderón-Zygmund Rule:

WE ARE HAPPY WITH A CUBE Q

if either

(a) $\boxed{\#(E \cap 3Q) \leq 1}$

or

(b) For some $\hat{a} < a$,

there exists an

$(\hat{a}, \tilde{C}_S(\bar{\bar{a}})\delta_Q, \tilde{C}_B(\bar{\bar{a}}))$- basis

for $\sigma_{\ell(a)-3}(z)$, all $z \in E \cap 3Q$.

BECAUSE E is <u>finite</u>,

Every small enough Q

satisfies (a),

So eventually we stop cutting.

We obtain a partition of

$3Q_0$ into Calderón-Zygmund

cubes Q_ν $(\nu = 1, \dots, \nu_{MAX})$.

The CZ cubes Q_ν

have

$$\boxed{\begin{array}{l}\text{GOOD GEOMETRY:}\\ \text{If } Q_\mu \text{ and } Q_\nu \text{ touch,}\\ \text{then their sidelengths differ}\\ \text{by at most a factor of 4.}\end{array}}$$

(NOT OBVIOUS, BUT TRUE.

WE SKIP THE PROOF.)

GOOD GEOMETRY

allows us to construct

a "Whitney partition of unity",

~~exact~~ adapted to the Q_ν :

$$\boxed{\begin{gathered} 1 = \sum_\nu \theta_\nu \quad \text{on } 3Q_0 \\ \theta_\nu \geq 0, \quad \operatorname{supp}(\theta_\nu) \subset (1.01) Q_\nu \\ |\partial^\alpha \theta_\nu| \leq C \delta_\nu^{-|\alpha|} \quad \text{for } |\alpha| \leq m \\ (\delta_\nu = \text{SIDELENGTH}(Q_\nu)). \end{gathered}}$$

BECAUSE THE Q_ν ARISE FROM SUCCESSIVE BISECTION, WE KNOW THAT

We are HAPPY with each Q_ν,

but

We are UNHAPPY with each Q_ν^+

($Q_\nu^+ = \left[\begin{array}{l}\text{THE CUBE WE BISECTED} \\ \text{TO ARRIVE AT } Q_\nu\end{array}\right]$)

BECAUSE WE ARE

UNHAPPY WITH Q_ν^+,

WE CANNOT HAVE

$\#(E \cap 3(Q_\nu^+)) \leq 1.$

So there exists

$$\boxed{x_\nu \in CQ_\nu \cap E.}$$

Fix such an x_ν, taking

$$\boxed{x_\nu \in 3Q_\nu \cap E \text{ if possible.}}$$

A SPECIAL EXCEPTION

If $x_0 \in E \cap 3Q_\nu$,

then we set $x_\nu := x_0$.

(RECALL, x_0 ARISES IN $Lip(Q_0, x_0, P_0)$)

WE HAVE NOW CUT UP $3Q_0$

INTO C-Z SUBCUBES Q_ν,

Constructed a Whitney partition

of unity $1 = \sum_\nu \theta_\nu$

adapted to the Q_ν,

and introduced base pts.

$x_\nu \in E \cap CQ_\nu$ for each ν.

NEXT, WE FIND A POLY.

$$P_\nu \in \Gamma_{\ell(a)-1}(x_\nu, M_0)$$

for each ν.

RECALL, WE ARE TRYING TO SOLVE

$LIP(Q_0, x_0, P_0)$,

WITH $P_0 \in \Gamma_{\ell(a)}(x_0, M_0)$.

By a basic property of the Γ_ℓ,

there exists

$P_\nu \in \Gamma_{\ell(a)-1}(x_\nu, M_0)$

s.t.

$$|\partial^\alpha (P_\nu - P_0)(x_0)| \leq M_0 |x_\nu - x_0|^{m-|\alpha|}$$

$$\leq C M_0 \delta_{Q_0}^{m-|\alpha|} \text{ (each } \alpha)$$

LET'S TRY USING

THAT

P_ν (EACH $\nu = 1, \ldots, \nu_{MAX}$).

NOTE THAT

$$P_2 = P_0$$

if $x_2 = x_0$.

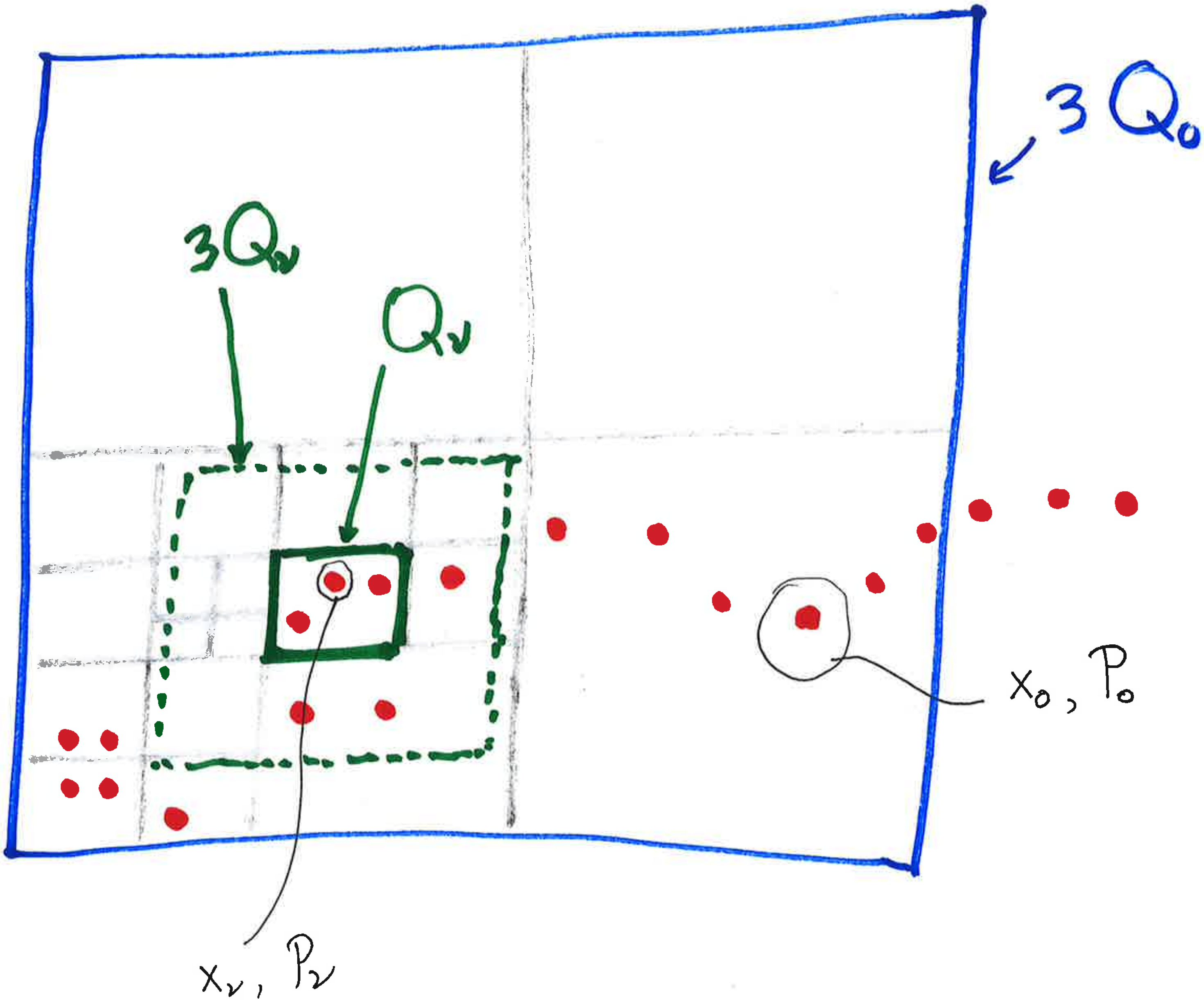
THE SITUATION IS NOW AS FOLLOWS:

$3Q_0$ is partitioned into Q_ν

We are happy with Q_ν, but unhappy with Q_ν^+.

We have found $x_\nu \in E \cap CQ_\nu$ and $P_\nu \in \Gamma_{\ell(a)-1}(x_\nu, M_0)$.

The Q_ν have GOOD GEOMETRY, and they give rise to $1 = \sum_\nu \theta_\nu$



THE C-Z CUBES Q_ν, WITH BASE PTS x_ν AND POLYS P_ν.

NEXT, WE FORMULATE

LOCAL INTERP. PROBLEMS

FOR THE Q_2.

Fix a Q_ν.

RECALL, WE ARE <u>HAPPY</u> WITH Q_ν.

So either

$\#(E \cap 3Q_\nu) \leq 1$ (TRIVIAL CASE)

or, for some $\hat{a} < a$,

the LOCAL INTERP. PROB.

$\mathcal{L}IP(Q_\nu, x_\nu, P_\nu)$

CARRIES THE LABEL $\hat{a}$

(THE NON-TRIVIAL CASE)

A TECHNICAL DETAIL:

HERE WE USE THE FACT THAT THE $\ell(a)$ ARE PICKED SO THAT $\ell(\hat{a}) \leq \ell(a) - 3$ WHENEVER $\hat{a} < a$.

Let's not dwell on the detailed verifications.

NEXT, WE PRODUCE LOCAL INTERPOLANTS.

For each Q_ν, we produce

$F_\nu \in C^m(3Q_\nu)$ s.t.

$$\boxed{\begin{array}{l} |\partial^\alpha F_\nu| \leq CM_0 \text{ on } 3Q_\nu \quad (|\alpha| = m) \\ J_x(F_\nu) \in \Gamma_0(x, CM_0), \text{ all } x \in E \cap (1.01)Q_\nu \\ J_{x_\nu}(F_\nu) = P_\nu \quad (\text{if } x_\nu \in 3Q_\nu) \end{array}}$$

IN THE NON-TRIVIAL CASE, WE CAN PRODUCE SUCH AN F_ν by APPLYING the MAIN LEMMA for $\hat{a}$ $(\hat{a} < a)$.

RECALL THAT $LIP(Q_\nu, x_\nu, P_\nu)$ carries the label $\hat{a}$, and

$$P_\nu \in \Gamma_{l(a)-1}(x_\nu, M_0) \subset \Gamma_{l(\hat{a})}(x_\nu, M_0)$$

IN THE TRIVIAL CASE,

WE HAVE TO PRODUCE

F_ν S.T.

$J_x(F_\nu) \in \Gamma_0(x, CM_0)$ for $x \in E \cap (1.01)Q_\nu$,

but $E \cap (1.01)Q_\nu$ contains

at most 1 point.

So we can produce F_ν in

the TRIVIAL CASE.

NOTE: There's a

SUPER-TRIVIAL CASE

in which $E \cap \partial Q_\nu = \emptyset$.

In that case, we can just take

$F_\nu = P_\nu$.

So far, we have cut $3Q_0$ into Calderón-Zygmund cubes Q_ν, produced LOCAL INTERPOLANTS F_ν that do what we want on $(1.01)Q_\nu$, and defined a Whitney partition of unity $1 = \sum_\nu \theta_\nu$ on $3Q_0$, with $\mathrm{supp}\,\theta_\nu \subset (1.01)Q_\nu$.

For our Grand Finale, we want to patch together the F_ν by setting

$$F = \sum_\nu \theta_\nu F_\nu .$$

We hope that F solves our LOCAL INTERPOLATION PROBLEM $LIP(Q_0, x_0, P_0)$.

OOPS!

IT DOESN'T WORK!

In order to estimate the $m^{\underline{th}}$ DERIVATIVES OF F,

WE REQUIRE

MUTUAL CONSISTENCY of the F_ν

i.e.

$$\boxed{\begin{array}{l} |\partial^\alpha(F_\mu - F_\nu)| \leq C M_0 \, \delta_\mu^{m-|\alpha|} \quad (|\alpha| \leq m) \\ \text{on } 3Q_\mu \cap 3Q_\nu \,, \\ \text{whenever } Q_\mu \text{ \& } Q_\nu \text{ touch.} \end{array}}$$

δ_μ = SIDELENGTH(Q_μ)

This issue arises already

in the proof of the

classical Whitney extension thm.

It's easy to estimate the derivatives of $F_\nu - P_\nu$,

so MUTUAL CONSISTENCY

is easily seen to be equivalent to the estimate

$$\boxed{\begin{array}{l} |\partial^\alpha (P_\mu - P_\nu)| \leq C M_0 \delta_\mu^{m-|\alpha|} \quad (|\alpha| \leq m-1) \\ \text{whenever } Q_\mu \text{ \& } Q_\nu \text{ touch.} \end{array}}$$

THAT MAY FAIL!

RECALL:

For each ν, we picked

$$P_\nu \in \Gamma_{\ell(a)-1}(x_\nu, M_0)$$

s.t.

$$|\partial^\alpha (P_\nu - P_0)(x_0)| \leq C\, \delta_{Q_0}^{m-|\alpha|} M_0$$

(all $|\alpha| \leq m-1$).

NOTE: P_ν, P_μ PICKED INDEPENDENTLY ($\mu \neq \nu$).

If $a \neq \emptyset$, then there is

ROOM TO MANEUVER

inside $\Gamma_{\ell(a)-1}(x_\nu, M_0)$.

There is a lot of <u>freedom</u>

in our choice of P_μ, P_ν.

Therefore, when $\mathcal{Q}_\mu$, $\mathcal{Q}_\nu$ TOUCH,

there is no reason to

expect that

P_μ & P_ν will be

MUTUALLY CONSISTENT.

THAT'S THE TROUBLE!

<u>SOLUTION</u> :

We will <u>EXPLOIT</u> the FREEDOM TO MANEUVER, to strengthen the basic property of T's used above.

WHAT WE ALREADY KNEW

Given $P_0 \in \Gamma_{\ell(a)}(x_0, M_0)$,

there exists $P_\nu \in \Gamma_{\ell(a)-1}(x_\nu, M_0)$

s.t.

$$\left|\partial^\alpha(P_\nu - P_0)(x_0)\right| \le M_0 \, |x_\nu - x_0|^{m-|\alpha|}$$

(all $|\alpha| \le m-1$)

WHAT WE DO NOW

Exploiting the

FREEDOM OF MANEUVER

inside $\Gamma_{\ell(a)}(x_0, M_0)$

provided by the LABEL a,*

we can now prove the following.

* (USING ALSO THE MONOTONICITY OF a)

Let $P_0 \in \Gamma_{l(a)}(x_0, M_0)$.

Then there exists

$P_\nu \in \Gamma_{l(a)-1}(x_\nu, CM_0)$

s.t.

$$|\partial^\alpha (P_\nu - P_0)(x_0)| \leq CM_0 \delta_{Q_0}^{m-|\alpha|}$$

(all $|\alpha| \leq m-1$)

AND ALSO

$\partial^\alpha (P_\nu - P_0) \equiv 0$ for $\alpha \in \mathcal{A}$.

WE MODIFY OUR PREVIOUS DISCUSSION BY PICKING EACH P_ν TO SATISFY THE ADDITIONAL CONDITION

$$\partial^\alpha(P_\nu - P_0) \equiv 0, \text{ all } \alpha \in \mathcal{A}.$$

Good News:

Our P_ν are now

Mutually Consistent

$$\boxed{\begin{gathered}|\partial^\alpha(P_\mu - P_\nu)(x_\nu)| \le CM_0\,\delta_\mu^{m-|\alpha|} \\ \text{for } |\alpha| \le m-1 \\ \text{whenever } Q_\mu, Q_\nu \text{ touch.}\end{gathered}}$$

That's because there isn't

TOO MUCH ROOM to MANEUVER

inside

$$\Gamma_{\ell(a)-1}(x_\nu, CM_0),$$

due to the fact that

WE DON'T LIKE Q_ν^+.

UNDERSTANDING THIS POINT IS THE HARDEST PART OF OUR PROOF.

LET'S JUST $\boxed{\text{DECLARE VICTORY}}$.

It works!

So we have completed our induction on $\mathcal{a}$ in the MONOTONIC CASE.

We already handled the NON-MONOTONIC CASE.

OUR INDUCTION IS COMPLETE.

WE HAVE PROVEN THE

MAIN LEMMA,

TODAY'S MAIN THM,

and all the RESULTS from LECTURE 3 re C^m INTERP. from $f: E \to \mathbb{R}$ for FINITE E.

WE GOT THROUGH IT!

Similar ideas
will underly our
later treatment of
INTERPOLATION & EXTENSION
in the setting of
SOBOLEV SPACES.

STAY TUNED!

P.S.

There is a shorter COORDINATE-FREE Proof of today's MAIN THM, due to

$\boxed{\begin{array}{l}\text{J. CARRUTH}\\ \text{A. FREI-PEARSON}\\ \text{A. ISRAEL}\\ \text{B. KLARTAG}\end{array}}$

That proof is shorter than the one sketched here by a factor of 2.

It can probably by adapted to other settings.

STAY TUNED !

THANK YOU!